From these results we can conclude that the dimensional heat-up time (more accurately, the e-folding time) is given by

$$t_h = Pr^{1/2}Ra^{1/4}\left(\frac{2L}{g\beta_0\Delta T}\right)^{1/2} \quad (33)$$

which is much less than a diffusion time (which scales as $Ra^{1/2}$). From equation (33), one can show that the heat-up time for an insulating air gap in a pane of thermal glass is of the order of seconds, for the liquid oxygen in a spacecraft fuel tank it is of the order of a few hours, for the liquified natural gas in typical land-based storage tanks it is of a few days, while for the earth's mantle, the heat-up time may be of the order of 10^9 yr. This last figure is very approximate as it depends upon the properties of the earth's mantle which are not accurately known. It does suggest, however, that the convection patterns in the earth's mantle, which are presumably responsible for continental drift, may not have yet reached steady state.

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LAMINAR AND TURBULENT HEAT TRANSFER BY NATURAL CONVECTION

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NOMENCLATURE

b, gap width between concentric cylinders or spheres;
C_l, C_t, constants, see (2) and (5);
D, diameter of cylinder or sphere;
D_i, D_o, inside and outside diameters respectively of concentric cylinder or sphere;
g, gravitational acceleration;
g_s, component of gravitational acceleration along the surface;
k, thermal conductivity;
k_{eff}, effective thermal conductivity of fluid in gap between concentric cylinders or spheres;
L, length of plate in flow direction;
$\overline{Nu}_s$, average Nusselt number based on the relevant dimension, s;
Pr, Prandtl number;
s, curvilinear distance along surface from stagnation point;
T_s, surface temperature;
T_∞, fluid temperature far from surface;
r, horizontal distance from axis of symmetry to point on surface;
Ra_s, Ra_r, Rayleigh number based on $(T_s - T_\infty)$ and dimension s and r, respectively;
β, thermal expansion coefficient;
Δ_l, Δ_t, laminar and turbulent conduction thickness;
κ, thermal diffusivity of fluid;
ν, kinematic viscosity of fluid.

INTRODUCTION

THERE is a striking similarity between the processes governing the growth of a liquid condensate film, and the growth of the inner region of a natural convection boundary layer. Exploitation of this analogy has enabled the authors to obtain a general approximate solution to a broad class of free convection problems, predicting heat-transfer rates in good agreement with experimental results. The detailed development of this method and the comparison of predictions and measurements for several problems will appear elsewhere [1]. This note is intended to draw the reader's attention to this method, and to summarize the results.

In the main, the method applies to the problem of assessing the heat transfer from the external surfaces of single two-dimensional or axisymmetric bodies immersed in an extensive fluid, although enclosure problems are also considered in [1]. The method is first developed and tested for the case where the flow is laminar over the entire surface of the body. It is then extended to the turbulent case. A simple criterion is then proposed to predict the extent of the surface subjected respectively to laminar and turbulent heat transfer.

LAMINAR HEAT TRANSFER

The velocity extremum in a free convection laminar boundary layer divides the flow into two regions: the inner region adjacent to the wall, and the outer region. A central premise of the present model is that, in the inner region, inertial forces are not important and energy transfer normal to the walls is by conduction only. It is also hypothesized that the fraction of the boundary layer's total buoyancy carried by the inner region is invariant with s. With these assumptions, the rate of growth of the thickness of the inner region is found to be completely fixed locally, in a manner similar to the growth of a condensate film. As a consequence, the heat transfer can be calculated directly by integrating along the surface. The resultant expression for the heat transfer is derived in [1]. Expressed in terms of the local conduction thickness (defined as that thickness of stagnant fluid offering the same resistance to heat transfer as that actually offered by the boundary layer) it is:

$$\Delta_l(s) = \frac{s}{C_l Ra_s^{1/4}} \left[\frac{\left\{ \frac{1}{s} \int_0^s \left(r^{4i} \frac{g_s}{g} \right)^{1/3} \mathrm{d}s \right\}^{1/4}}{(r^i g_s/g)^{1/3}} \right] \tag{1}$$

C_l is a function only of the Prandtl number and is given approximately by:

$$C_l = 0{\cdot}48 \left(\frac{Pr}{0{\cdot}861 + Pr} \right)^{1/4}. \tag{2}$$

The value of i is zero for two-dimensional bodies and unity for axisymmetry ones. An expression, similar to (1) can be derived for the average conduction thickness, $\bar{\Delta}_l$; it is given in [1].

Equation (1) applies to the special case where T_s and T_∞ are constant. A more general equation, accounting for variations in T_s and T_∞, is given in [1]. When Δ_l is small compared to the body dimensions, the local heat flux per unit area, q/A, is computed from (1) using locally the equation $(q/A) = k(T_s - T_\infty)/\Delta_l$. When Δ_l becomes large, curvature effects are approximately accounted for by surrounding the body with a stagnant fluid layer of (variable) thickness given by $\Delta_l(s)$ as computed from equation (1), and solving the resultant conduction problem. Usually an approximate "quasi-one dimensional" solution to this problem is sufficient.

TURBULENT HEAT TRANSFER

An equation for the local conduction thickness for turbulent flow is proposed in [1]. It is:

$$\Delta_t = \frac{1}{C_t A(\phi)} \left(\frac{\nu\kappa}{g\beta(T_s - T_\infty)} \right)^{1/3} \tag{3}$$

where

$$A(\phi) = \cos^{1/3}\phi \quad \text{for} \quad 19° < \phi \leqslant 90° \tag{4a}$$

$$A(\phi) = 0{\cdot}71 \sin^{1/3}\phi \quad \text{for} \quad -90° \leqslant \phi < 19° \tag{4b}$$

$$C_t = 0{\cdot}14 Pr^{0{\cdot}084} \tag{5}$$

and ϕ is the local angle of the surface from the vertical. ϕ is positive for either an upward facing heated surface or a downward-facing cooled surface. The above expressions for Δ_l and Δ_t are similar. In both cases the body shape is represented exclusively through one term, namely $A(\phi)$ in the turbulent case, and the squared-bracketed part of (1) in the laminar case. Similarly the Prandtl number dependence is represented through the terms C_t and C_l. In both cases the remaining term represents the relevant length scale. For turbulent flow the important length scale is independent of s and is equal to the "thermal length", $Y_t = (\nu\kappa/g\beta\Delta T)^{1/3}$; for laminar flow it is $Y_t . (s/Y_t)^{1/4}$.

LAMINAR AND TURBULENT HEAT TRANSFER

When laminar and turbulent heat transfer occur simultaneously on different portions of a body, a rule is needed to establish the extent of the surface covered by each. The following simple procedure, based entirely on *local* quantities is proposed in [1]. It is:

$$\left. \begin{array}{ll} \frac{4}{3}\Delta_l \leqslant \Delta_t, & \text{equation (1) is valid} \\ \frac{4}{3}\Delta_l > \Delta_t, & \text{equation (3) is valid.} \end{array} \right\} \tag{6}$$

The "transition" laminar conduction thickness is therefore given by $\Delta_l = 3/4\Delta_t$. Substituting equations (1) and (3) into this condition and solving for Ra_s yields a "transition" Rayleigh number, denoted by Ra_s^t. Ra_s^t is plotted as the solid curves in Fig. 1 for the special case of a flat tilted plate. The data [2, 3] are measurements of Ra_s at the onset of turbulence, denoted by $Ra_{c,s}$. The agreement of these data with the above criterion lends support to the proposed rule.

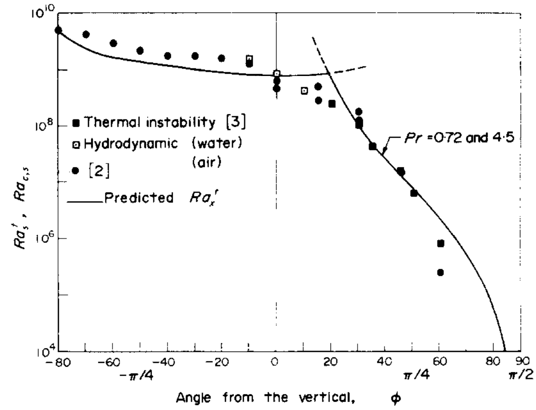

FIG. 1. Measurements of critical Rayleigh number, $Ra_{c,x}$, and Rayleigh corresponding to $\Delta_l = 4/3\Delta_t$, Ra_x^t.

RESULTS

The above model is applied to a number of specific geometries in [1]. The following summarizes the findings.

Heat transfer from a circular cylinder

For this geometry $i = 0$ and equation (1) leads to:

$$\overline{Nu}_D = \frac{2}{\ln(1 + 1{\cdot}94/C_l Ra_D^{1/4})}. \tag{7}$$

To obtain this result the curvature correction has been made. For turbulent flow everywhere on the cylinder, equation (2) yields:

$$\overline{Nu}_D = 0{\cdot}72 C_t Ra_D^{1/3}. \tag{8}$$

The results for laminar and turbulent flow on different portions of the cylinder, obtained from equation (6) are given in [1]. The equations are in good agreement with the measurements reported in [4, 5].

Laminar heat transfer from a sphere

Applying equation (1) to a sphere ($i = 1$), and accounting for curvature effects, leads to

$$\overline{Nu}_D = 2 + 1{\cdot}17 C_l Ra_D^{1/4}. \tag{9}$$

Heat transfer between concentric cylinders and spheres

For concentric cylinders, each cylinder is considered to be surrounded by a stagnant fluid layer of thickness given by (1). The fluid between the stagnant layers is assumed to vary in temperature only in the circumferential direction. Since this circumferential temperature variation is unknown, an unknown constant arises in the solution which must be determined from experiment. The final equation for the heat flow, expressed in terms of the effective thermal conductivity, k_{eff}, is:

$$\frac{k_{\text{eff}}}{k} = 0{\cdot}80 C_l \frac{\ln(D_o/D_i)}{b^{3/4}\{D_i^{-3/5} + D_o^{-3/5}\}^{5/4}} \cdot Ra_b^{1/4}. \tag{10}$$

If k_{eff}/k from (10) is smaller than unity, $k_{\text{eff}}/k = 1{\cdot}0$ is to be used. The data of Grigull and Hauf [7] and Beckman [8] for air are correlated in Fig. 2 using the parameters in (10). Data for other Prandtl numbers are also well

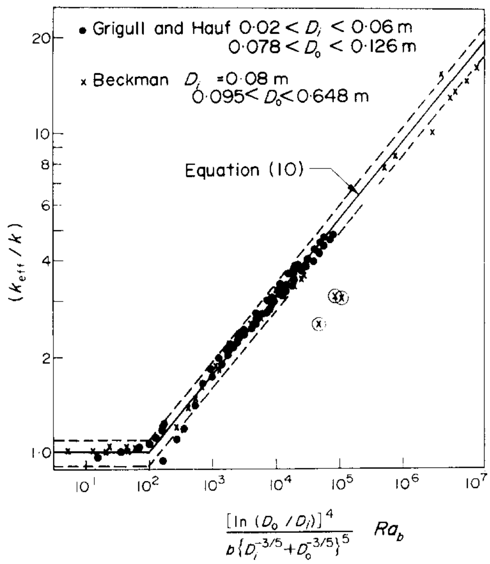

FIG. 2. Correlation of data for heat transfer across the gap between concentric cylinders. Circled data points appear to contain errors in tabulation.

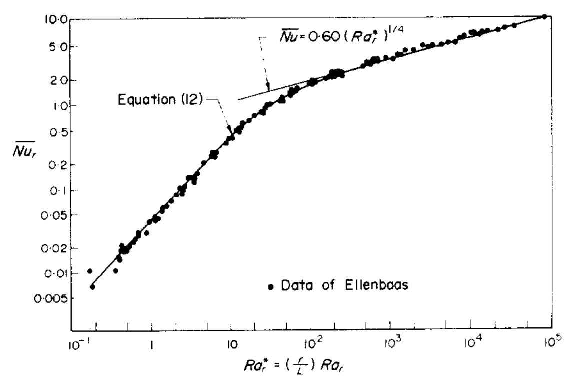

FIG. 3. Correlation of data for a cooling slot with isothermal walls at the same temperature.

correlated by this expression. A similar analysis for concentric spheres yields:

$$\frac{k_{eff}}{k} = 1{\cdot}54\,C_l \frac{b^{1/4}}{D_o D_i (D_i^{-7/5} + D_o^{-7/5})^{5/4}} \cdot Ra_b^{1/4}. \qquad (11)$$

As before, if k_{eff}/k from (11) is less than unity, a value of 1·0 is to be used. The data of Bishop, Powe *et al.* [9] are well correlated by this equation for a wide range of geometries for both air and oils.

Heat transfer from a cooling slot

The heat transfer to the fluid flowing by natural convection between two parallel and vertical plates is found to be given by:

$$\overline{Nu}_r = \tfrac{4}{3} C_l (Ra_r^*)^{1/4} \sum_{n=1}^{\infty} (-1)^{n+1} \times \frac{3}{(4n-1)(n-1)!}\left(\frac{C_3}{Ra_r^*}\right)^{n-1} \qquad (12a)$$

$$\to \frac{\Gamma(7/4)}{C_3^{3/4}} Ra_r^* \quad \text{as} \quad Ra_r^* \to 0 \qquad (12b)$$

r is the hydraulic diameter of the passage (twice the spacing between the plates), L is the length of the cooling passage, $\Gamma(7/4)$ is a gamma function, and $Ra_r^* = (r/L)Ra_r$. When the plates are isothermal and identical in temperature, the constant C_3 was found to be $[32\Gamma(7/4)C_l]^{4/3}$; if one of the plates is adiabatic C_3 is $[16\Gamma(7/4)C_l]^{4/3}$. A slight alteration of the constant C was made to improve the agreement with Elenbaas' data [10] for widely spaced plates. A comparison of equation (12) and Elenbaas' data is given in Fig. 3.

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